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Optimal Steiner hull algorithm

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Abstract

Given a set Z of n points in the plane, we consider the problem of finding the Steiner hull for Z which is a non-trivial polygon containing every Euclidean Steiner minimal tree for Z. We give an optimal $\Theta(n \log n)$ time and $\Theta(n)$ space algorithm exploiting a Delaunay triangulation of Z. If the Delaunay triangulation is given, the algorithm requires linear time and space. Furthermore, we argue that the uniqueness argument for the $O(n^3)$ time Steiner hull algorithm given in [4] is incorrect, and we give a correct uniqueness proof.

Keywords: Algorithms; Computational geometry; Steiner tree problem; Steiner hull; Delaunay triangulation

1. Introduction

The *Euclidean Steiner tree problem* (ESTP) is as follows.

- *Given*: A set Z of n points, also referred to as terminals, in the plane.
- *Find*: The shortest tree, referred to as the *Euclidean Steiner minimal tree* (ESMT) spanning the terminals.

Edges of the ESMT can meet anywhere in the plane. Exactly three edges must meet at locations other than terminals making 120° with each other. At most three edges can meet at terminals. The reader is referred to [4] for other properties of ESMTs.

Every ESMT for Z must be in the convex hull for Z, denoted by $CH(Z)$. We address the problem of finding a smaller region containing every ESMT for Z. One such region is known in the literature as the *Steiner hull* for Z, and is denoted by $SH(Z)$. Its precise definition will be given at the end of Section 2.

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It is well-known that if the number of terminals on $SH(Z)$ increases, then the ESTP is easier to solve [4]. If $SH(Z)$ is not simple, then the ESTP decomposes into smaller ESTPs.

Our motivation to investigate Steiner hulls stems from the fact that we can today find ESMTs for fairly large problem instances with 1.000–2.000 terminals [6]. Some additional improvements suggested in [1] and already implemented in [7] permit to solve problem instances with up to 10.000 terminals. For such problem instances, a fast bounding algorithm can be of interest (although it is not a bottleneck of the ESMT algorithm). Algorithms to determine $SH(Z)$ applied until now require $O(n^3)$ time and $O(n)$ space.

2. Definitions and basic properties

Let P_Z denote a polygon with a subset of terminals as its vertices, and such that P_Z contains every ESMT for Z. Let $CH(Z)$ be the initial P_Z. Consider a pair of terminals z_u and z_v belonging to P_Z. Let $P_Z(z_u, z_v)$ denote the polygonal chain between z_u and z_v with the interior of P_Z to the right when moving from z_u toward z_v.

Let z_iz_j be an edge on the boundary of P_Z, i.e. $z_iz_j = P_Z(z_i, z_j)$ with the interior of P_Z to the right of z_iz_j when looking from z_i toward z_j. A replacement of z_iz_j by a pair of edges z_iz_k and z_kz_j, $z_k \in Z\backslash\{z_i, z_j\}$, denoted by $z_iz_j \to z_iz_kz_j$, is said to be *legal* if

- $\triangle z_iz_jz_k$ is contained in P_Z,
- $\triangle z_iz_jz_k$ contains no terminals other than its corners,
- $\angle z_iz_kz_j \geqslant 120°$.

Each legal replacement $z_iz_j \to z_iz_kz_j$ corresponds to a triangle with one of its sides, called the *base*, being replaced by the other two sides and with z_k as a new polygon-vertex. Note that a legal replacement may result in a non-simple polygon.

Lemma 1 [3]. *If there is a legal replacement $z_iz_j \to z_iz_kz_j$ of the edge z_iz_j of P_Z, then the reduced polygon also contains every ESMT for Z.*

Proof. Consider the triangle $\triangle z_iz_kz_j$ and an ESMT T for Z. Since the edges of T make 120° with each other at Steiner points, $\triangle z_iz_kz_j$ cannot contain a Steiner point. One of the three edges incident with such a Steiner point would end outside of P_Z (Fig. 1(a)).

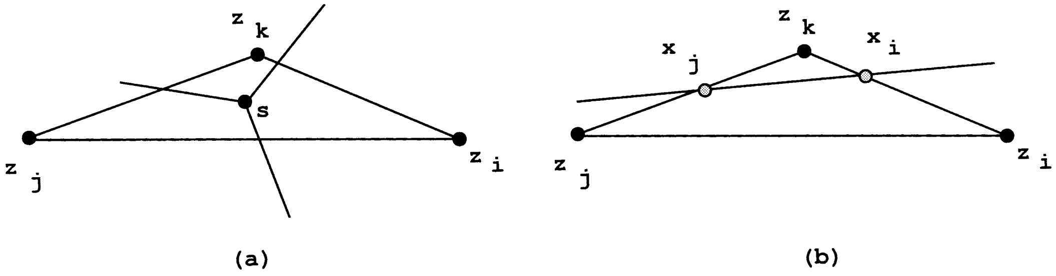

Fig. 1. $\triangle z_iz_jz_k$ contains no ESMT.

Suppose that $\triangle z_i z_k z_j$ contains an edge of T. Assume that this edge crosses $z_i z_k$ in x_i (possibly $x_i = z_i$), and crosses $z_k z_j$ in x_j (possibly $x_j = z_j$) as shown in Fig. 1(b). Remove $x_i x_j$ from T. One of the resulting subtrees contains z_k. Reconnect the subtrees by introducing an edge from z_k to either x_i or x_j depending on which one does not generate a cycle. A shorter tree is obtained, a contradiction. $\square$

A simple algorithm follows from Lemma 1: Let $P_Z = CH(Z)$. Apply legal replacements to the edges of P_Z for as long as possible (e.g., in depth-first, clockwise order, beginning with a fixed edge of $CH(Z)$). In the remainder of this section we will show that the order in which legal replacements are applied is immaterial; *every* maximal sequence leads to the same final P_Z. The proof of this fact is more complicated than the proof given in [4, p. 11]. The latter is based on the following (incorrect) observation: If $z_i z_j \to z_i z_h z_j$ and $z_i z_j \to z_i z_k z_j$ are both legal replacements, then there are *consecutive* legal replacements that will introduce z_h directly followed by z_k or vice versa. This is not true as the appearance of z_h on P_Z may in fact postpone the appearance of z_k on P_Z as shown in Fig. 2: When $z_i z_j \to z_i z_h z_j$ is applied first, $z_h z_j \to z_h z_k z_j$ is illegal since $\triangle z_h z_k z_j$ contains z_u. Similarly, if $z_i z_j \to z_i z_k z_j$ is applied first, then $z_i z_k \to z_i z_h z_k$ is illegal since $\triangle z_i z_h z_k$ contains z_u.

Lemma 2. *If there is a legal replacement $z_i z_j \to z_i z_k z_j$, then z_k will be a vertex of every polygonal chain $P_Z(z_i, z_j)$ obtained by any maximal sequence of legal replacements applied to $z_i z_j$ in P_Z.*

Proof. Consider a maximal sequence S of legal replacements applied to a boundary edge $z_i z_j$ of P_Z. After each legal replacement, there is an edge $z_u z_v$ of the current boundary between z_i and z_j that penetrates $\triangle z_i z_k z_j$. Furthermore, $120^\circ \leqslant \angle z_i z_k z_j < \angle z_u z_k z_v$.

Consider the situation when all legal replacements of S have been applied, and the boundary of P_Z between z_i and z_j does not contain z_k. Let x_u denote the intersection of $z_u z_v$ with $z_i z_k$ (possibly $x_u = z_i$), and let x_v denote the intersection of $z_u z_v$ with $z_j z_k$ (possibly $x_v = z_j$) as shown in Fig. 3. The replacement $z_u z_v \to z_u z_k z_v$ can be illegal only because $\triangle z_u z_k z_v$ contains a terminal. It is then always possible to pick a terminal z in $\triangle z_k z_u z_v$ such that $\triangle z_u z z_v$ contains no other terminal, and $\angle z_u z z_v \geqslant 120^\circ$. This implies that $z_u z_v \to z_u z z_v$ is a legal replacement, contradicting the maximality of S. $\square$

Let z_k denote a terminal such that $z_i z_j \to z_i z_k z_j$ is legal, and the projection of z_k onto $z_i z_j$ is as close to z_i as possible. Such a legal replacement is referred to as *canonical.* A maximal sequence of legal replacements is said to be *canonical* if all its legal replacements are canonical.

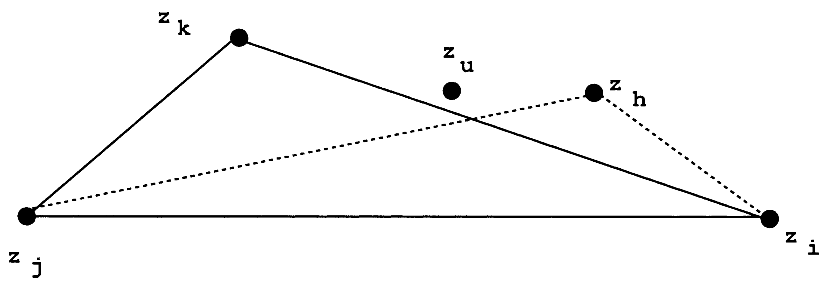

Fig. 2. Counterexample.

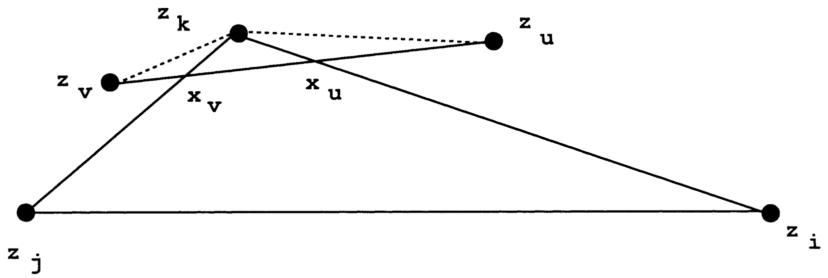

Fig. 3.

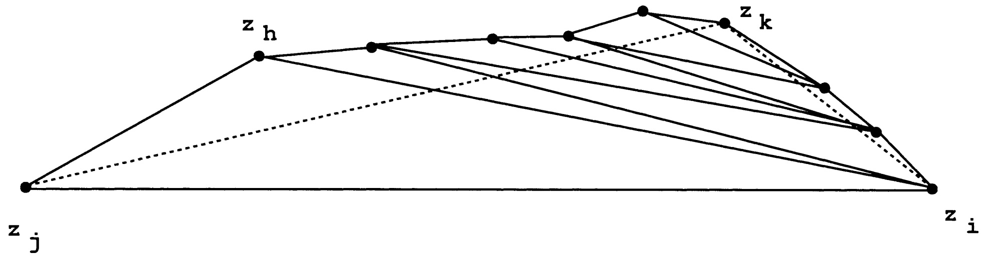

Fig. 4. Legal replacements between z_h and z_k.

Lemma 3. *Polygonal chain $P_Z(z_i, z_j)$ obtained by any maximal sequence of legal replacements can also be obtained by a canonical sequence.*

Proof. Let S be any non-canonical maximal sequence of legal replacements. Assume that the first legal replacement in S, denoted by $z_iz_j \to z_iz_hz_j$ is not canonical. We will show that $z_iz_j \to z_iz_hz_j$ and some of subsequent legal replacements in S can be substituted by the canonical legal replacement $z_iz_j \to z_iz_kz_j$ followed by some appropriately chosen legal replacements. Furthermore, S and the new sequence terminates with the same polygonal chain. By repeating this process sufficiently many times, a canonical maximal sequence terminating with the same polygon as S will be obtained.

According to Lemma 2, z_k will sooner or later appear on $P_Z(z_i, z_j)$. Let S' denote the subsequence of S beginning with $z_iz_j \to z_iz_hz_j$, and ending with the legal replacement introducing z_k on the boundary of P_Z between z_i and z_j.

Consider the sequence of triangles between z_iz_j and z_k intersected by z_jz_k as shown in Fig. 4. Let S'' denote the corresponding legal replacements (preserving their order in S'). Let $\overline{S''} = S' \backslash S''$ denote the remaining legal replacements of S' (preserving their order). It is obvious that S'' followed by $\overline{S''}$ generates the same boundary as S'. Hence, we can assume that S'' is a prefix of S'.

Consider the triangulated polygon P generated by S''. We will show that there exists another triangulation of P such that all triangles correspond to legal replacements, and such that $z_iz_j \to z_iz_kz_j$ is the first legal replacement.

Add the edges z_iz_k and z_kz_j. Suppose that $P_Z(z_k, z_j)$ has at least one intermediate terminal. Let z_h denote the predecessor of z_j. If $\triangle z_kz_hz_j$ contains no other terminals of $P_Z(z_k, z_j)$, then $z_kz_j \to z_kz_hz_j$ is a legal replacement. This follows from the fact that there is a terminal z_c in $P_Z(z_i, z_k)$ such that $z_cz_j \to z_cz_hz_j$ is a legal replacement in S'' (in Fig. 4, $z_c = z_i$). In particular, z_cz_h crosses z_kz_j. Hence, $\angle z_kz_hz_j > \angle z_cz_hz_j \geqslant 120°$. Repeat this procedure for $P_Z(z_k, z_h)$ if it has intermediate terminals.

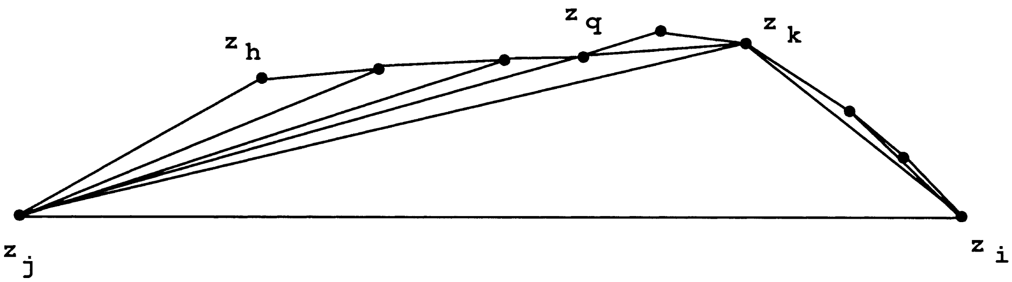

Fig. 5. Sequence of legal replacements generating the same boundary as in Fig. 4 but with z_k added before z_h.

If $\triangle z_k z_h z_j$ contains one or more terminals of $P_Z(z_k, z_j)$, then it is always possible to find another terminal z_q on $P_Z(z_k, z_j)$ in the interior of $\triangle z_k z_h z_j$ such that $\triangle z_k z_q z_j$ contains no other terminals. Now, $z_k z_q z_j > z_k z_h z_j > 120°$ implies that $z_k z_j \to z_k z_q z_j$ is a legal replacement. Repeat this procedure for $P_Z(z_k, z_q)$ and $P_Z(z_q, z_j)$ if they have intermediate terminals.

The same procedure (with straightforward modifications) applies to $P_Z(z_i, z_k)$. Once this procedure stops, the triangulation corresponds to the sequence of legal replacements, and it has $z_i z_j \to z_i z_k z_j$ as the first legal replacement. □

Theorem 1 [4]. *The polygon P_Z obtained by any maximal sequence of legal replacements, beginning with $CH(Z)$, is unique.*

Proof. Follows immediately from the fact that there is only one maximal canonical legal replacement sequence of each edge of $CH(Z)$. □

This unique polygon is referred to as the *Steiner hull* for Z, and is denoted by $SH(Z)$.

3. Algorithm

In this section we show that the pool of $O(n^3)$ legal replacements can be reduced to $O(n)$ legal replacements corresponding to triangles of a Delaunay triangulation.

Consider a Delaunay triangulation of Z, denoted by $DT(Z)$. See [2] for definitions and basic properties of Delaunay triangulations. Let $P_Z = CH(Z)$. Note that $CH(Z)$ is the exterior face of $DT(Z)$. Any triangle $\triangle z_i z_k z_j$ with $z_i z_j$ belonging to P_Z and with $\angle z_i z_k z_j \geqslant 120°$ defines a legal replacement $z_i z_j \to z_i z_k z_j$. This follows directly from the fact that the circle circumscribing $\triangle z_i z_j z_k$ in $DT(Z)$ contains no other terminals [2]. We refer to such legal replacements as *DT-based*.

The algorithm applies DT-based legal replacements to each of the edges of P_Z for as long as possible.

Lemma 4. *The algorithm terminates with $P_Z = SH(Z)$.*

Proof. Suppose that the algorithm terminates with $P_Z \neq SH(Z)$. Since $SH(Z)$ can be obtained from $CH(Z)$ by any maximal sequence S of legal replacements, assume that DT-based legal replacements applied to $P_Z = CH(Z)$ form a prefix of S. Consider the next legal replacement $z_i z_j \to z_i z_k z_j$ in S. Consider the triangle of $DT(Z)$ based on $z_i z_j$. Let z_l denote its third corner. Note that $z_l \neq z_k$. The

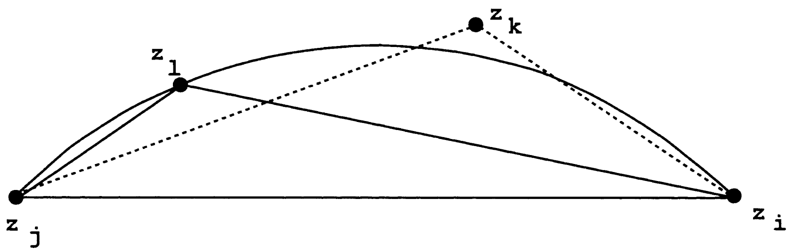

Fig. 6.

circle circumscribing $\triangle z_i z_l z_j$ is empty since $\triangle z_i z_l z_j$ is in $DT(Z)$. In particular, it cannot contain z_k. This implies that $\angle z_i z_l z_j \geqslant \angle z_i z_k z_j \geqslant 120°$. Hence, $z_i z_j \to z_i z_l z_j$ is a DT-based legal replacement, a contradiction. □

Lemma 5. *The algorithm requires* $\mathrm{O}(n \log n)$ *time and* $\mathrm{O}(n)$ *space.*

Proof. The determination of $DT(Z)$ requires $\Theta(n \log n)$ time and $\Theta(n)$ space. During each iteration, an edge of P_Z is examined. If this edge is replaced, two new edges appear on P_Z. Each of them is recursively examined (unless it already is an edge on P_Z). The total number of replaced edges is $\mathrm{O}(n)$. Examined but not replaced edge can occur only if it is on $CH(Z)$ or if it was introduced after an edge has been replaced. There are $\mathrm{O}(n)$ edges on the initial $P_Z = CH(Z)$. At most two edges are examined after each replacement. Hence, also in this case, at most $\mathrm{O}(n)$ edges is examined. It follows that the determination of $DT(Z)$ dominates the time complexity of the algorithm if the edges can be examined and replaced in $\mathrm{O}(1)$ time.

If $DT(Z)$ is represented as a doubly connected edge list, $\mathrm{O}(n)$ space is needed. Both the examination (requiring the verification whether or not one of the interior angles of currently examined triangle is 120° or more) and the replacement of an edge requires $\mathrm{O}(1)$ time. □

Theorem 2. *The Steiner hull algorithm is optimal.*

Proof. $SH(Z)$ is a polygon with non-intersecting edges. It is not simple since it can have pairs of non-consecutive edges sharing a point.

If $SH(Z)$ is given, $CH(Z)$ can be determined in $\mathrm{O}(n)$ time. This follows from the fact that $CH(Z) = CH(SH(Z))$. Convex hull of vertices of a polygon with non-intersecting edges can be determined in $\mathrm{O}(n)$ time [5]. Hence, any Steiner hull algorithm requires $\Omega(n \log n)$ time. □

4. Concluding remarks

We presented an optimal $\Theta(n \log n)$ time and $\Theta(n)$ space algorithm for the determination of the Steiner hull for the set of terminals in the plane. A natural generalization is that of determining geodesic Steiner hull for terminals being vertices of a simple polygon. Alternatively, the terminals may be allowed to be in the interior of the simple polygon (with or without holes).

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